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An approach to the equivalence theorem by the Slavnov-Taylor identities

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Abstract: We discus the Equivalence Theorem (ET) in the BRST formalism. In particular the Slavnov-Taylor (ST) identities, derived at the formal level in the path-integral approach, are considered at the quantum level and are shown to be always anomaly free. Some discussion is devoted to the transformation of the fields; in fact the existence of a local inverse (at least as a formal power expansion) suggests a formulation of the ET, which allows a nilpotent BRST symmetry. This strategy cannot be implemented at the quantum level if the inverse is non-local. In this case we propose an alternative formulation of the ET, where, by using Faddeev-Popov fields, this difficulty is circumvented. In fact this approach allows the loop expansion both in the original and in the transformed theory. In this case the algebraic formulation of the problem can be simplified by introducing some auxiliary fields. The auxiliary fields can be eliminated by using the method of Batalin-Vilkovisky. We study the quantum deformation of the associated ST identity and show that a selected set of Green functions, which in some cases can be identified with the physical observables of the model, does not depend on the choice of the transformation of the fields. The computation of the cohomology for the classical linearized ST operator is performed by purely algebraic methods. We do not rely on power-counting arguments. In general the transformation of the fields yields a non-renormalizable theory. When the equivalence is established between a renormalizable and a non-renormalizable theory, the ET provides a way to give a meaning to the last one by using the resulting ST identity. In this case the Quantum Action Principle cannot be of any help in the discussion of the ET. We assume and discuss the validity of a Quasi Classical Action Principle, which turns out to be sufficient for the present work. As an example we study the renormalizability and unitarity of massive QED in Proca's gauge by starting from a linear Lorentz-covariant gauge.

Keywords: Renormalization Regularization and Renormalons, BRST Quantization, BRST Symmetry.

Contents

1. Introduction

The equivalence theorem (ET) has a long history [1]–[7], conventionally starting from the alleged physical equivalence of the pseudoscalar and pseudovector coupling in the pion-nucleon system. Since then the significance of the statement has been enlarged and more sophisticated techniques have been used to prove the theorem. The ET is still a subject of some interesting researches [8]–[11]. The theorem follows from the fact that one can pass from a given theory to a second one by a change of the fields in the lagrangian. This is well-known, but to our knowledge it has not been considered in full generality.

Moreover all known proofs of the ET are based on the validity of the Quantum Action Principle (QAP), which is valid for power-counting renormalizable theories. In the pioneer papers it is assumed either that there is only one derivative for each field in the vertex (BPHZ framework) [12, 13] or that the Feynman propagators are standard ($(p^2 - m^2)^{-1}$ for bosons and $(\not{p} - m)^{-1}$ for fermions) and there are no derivative vertices (dimensional regularization) [14]. In [15] the proof of QAP is extended to the massless case for a power-counting renormalizable theory.

Thus typically the proofs rely on the power-counting renormalizability in order to avoid lagrangian terms causing the theory to become non-renormalizable. A generalization of these results is often made for non-renormalizable theories, i.e. it is assumed a straightforward extension of the results of the QAP to non-renormalizable theories.

In perturbative QFT, the renormalization problems to be solved result into rather disturbing complexity. It is a goal of this note to gather the cumulative experience of the past two decades to propose a step to simplify the problem by introducing the concept of Quasi Classical Action Principle (QCAP) ([16] and appendix A) and the linear Slavnov Taylor (ST) identities.

One of the major advances in Quantum Field Theory (QFT) has been the introduction of Feynman's path-integral quantization. Starting from a classical action $S[\Phi]$ (in our notations we use only Lorentz scalars, although the formalism can be applied to more general cases), Green functions are perturbatively constructed according to Gell-Mann and Low's expansion formula [17]. *Perturbative QFT* means here that the quantities of interest are computed order by order in $\hbar$ which is the loop expansion parameter, without any attempt to give a meaning to the complete sum. In terms of Feynman's path-integral representation the formal expression of the perturbative expansion of the expectation value of the product of local operators $\mathcal{O}_i(\Phi(x_i))$ is

$$\left\langle T\left(\prod_{i=1}^{n}\mathcal{O}_i(\Phi(x_i))\right)\right\rangle \propto \int[\mathcal{D}\Phi]\left(\prod_{i=1}^{n}\mathcal{O}_i(\Phi(x_i))\right)e^{iS[\Phi]}\,. \tag{1.1}$$

The properties of Feynman's path integral can be proven only at the formal level. Thus the power of the formalism is to provide relations among the Green functions which often need a more sound formulation.

It was early recognized by Becchi, Rouet, Stora and Tyutin (BRST) in the works [18, 19, 20] on BRST invariance of gauge theories and on the models with broken rigid symmetries [21], that the use of the Quantum Action Principle leads to the possibility of a fully algebraic proof of the renormalizability of a theory characterized by a set of local or rigid invariances.

Since then it has become a common paradigm in QFT to write Ward identities by using the path-integral and then use powerful methods of perturbation theory to prove them (as BPHZ renormalization scheme, BRST symmetry, etc.).

Let the classical action $S[\Phi]$ be a local functional of the field Φ (and its derivative) with non degenerate quadratic part. Let $\{\Gamma[\Phi,\beta]\}$ be the set of quantum extensions of $S[\Phi]$. $\Gamma[\Phi,\beta]$ is a formal power series in $\hbar$, the loop counting parameter, where the β are introduced in order to generate the Green functions in eq. (1.1). Their connected part is obtained by differentiating the generating functional given by the Legendre transform of the quantum action Γ with respect to the field Φ:

$$W[J,\beta] = \Gamma[\Phi,\beta] + \int d^4x \; J(x)\Phi(x) \,.$$

In renormalizable field theories the ambiguities in the definition of Γ are removed by power-counting and normalization condition. Now suppose that we wish to parameterize the classical action in a different way, i.e. we wish to perform the following field redefinition:

$$\Phi \to \Phi = \Phi(\varphi;\rho) \qquad \text{with} \qquad \Phi(\varphi;0) = \varphi\,; \tag{1.2}$$

with $\rho = \{\rho_i\}$ external classical fields parameterizing the transformation. How is eq. (1.1) affected by this field redefinition? In particular, do the physical observables depend on the choice of the field parameterization?

If one regards the above transformation as a variable substitution in the path integral formulation of eq. (1.1) then the ET becomes

$$\int [\mathcal{D}\Phi] \left(\prod_{i=1}^{n} \mathcal{O}_i \left(\Phi(x_i)\right) \right) e^{iS[\Phi]} = \int [\mathcal{D}\varphi] \left(\prod_{i=1}^{n} \mathcal{O}_i \left(\Phi(\varphi;\rho)(x_i)\right) \right) e^{iS[\Phi(\varphi;\rho)]+iD(\varphi;\rho)} \,, \tag{1.3}$$

where $D(\varphi;\rho)$ originates from the jacobian of the field redefinition, and $\mathcal{O}$ is any local operator. We denote this formulation of the ET as the Strong Equivalence Theorem (SET). In order to give meaning to the r.h.s. of eq. (1.3) a set of external sources will be necessary as for instance a term involving the new field φ and the Faddeev-Popov ghosts which will be introduced in order to take into account the determinant.

It has been noticed in [13] that a simpler formulation of the ET can be given if one neglects the jacobian of the transformation. In this case the ET takes the form

$$\int [\mathcal{D}\Phi] \left(\prod_{i=1}^{n} \mathcal{O}_i \left(\Phi(x_i)\right) \right) e^{iS[\Phi]} = \int [\mathcal{D}\varphi] \left(\prod_{i=1}^{n} \mathcal{O}_i \left(\Phi(\varphi;\rho)(x_i)\right) \right) e^{iS[\Phi(\varphi;\rho)]} \,, \tag{1.4}$$

which we call the Weak Equivalence Theorem (WET). This has been considered in different ways by previous authors [11, 13, 14]. In some cases (see discussion below about ρ expansion and loop expansion) it yields a set of local (counter)terms which can be reabsorbed by a redefinition of the action. However some particularly interesting cases (those with non-local inverse transformation) cannot be dealt with in this way.

The WET amounts to saying that Φ and φ are physically equivalent interpolating fields in the LSZ formalism.

The content of eq. (1.4) can be discussed in the framework of BRST theory where the transformations are chosen in such a way that $\Phi(\varphi;\rho)$ is an invariant [9, 11]

$$s\Phi(\varphi;\rho) = 0 \,. \tag{1.5}$$

The transformation of φ and ρ_k should then be

$$s\varphi = -\left(\frac{\delta\Phi}{\delta\varphi}\right)^{-1} \frac{\delta\Phi}{\delta\rho_k}\,\theta_k \,, \qquad s\rho_k = \theta_k \,, \qquad s\theta_k = 0 \,. \tag{1.6}$$

This transformation is required to be formally local as a power series in ρ_k.

We now consider the Strong Equivalence Theorem (SET). It states that the Green functions generated by the functional

$$Z[J] \propto \int [\mathcal{D}\Phi] \exp\left(iS[\Phi] + i\int d^4x\, J\Phi\right) \tag{1.7}$$

are related to those obtained from

$$\tilde{Z}[J,K;\rho] \propto \int [\mathcal{D}\varphi] \exp\left(iS[\Phi(\varphi;\rho)] + iD(\varphi;\rho) + i\int d^4x\, J\Phi(\varphi;\rho) + i\int d^4x\, K\varphi\right), \tag{1.8}$$

where $D(\varphi;\rho)$ originates from the jacobian of the field transformation. Roughly speaking, eq. (1.3) could be summarized by saying that the Green functions are insensitive to the fields reparameterization: once the classical action is fixed, the parameterization of the fields which we use in the path-integral is a matter of convenience.

As in the WET case, the content of eq. (1.3) can be discussed in the framework of BRST theory. However this problem has no straightforward solution. We find more convenient to introduce a set of auxiliary fields, which allows a linearization of the ST identities. We call it the *off-shell* formalism. Eventually we shall provide also the BRST transformations relevant for the *on-shell* theory which yields the generating functional in eq. (1.8).

One should remark that even if the l.h.s. of eq. (1.3) refers to a renormalizable theory, the action in the r.h.s. could be not-renormalizable and therefore the question will soon arise whether it is possible to constrain the renormalization ambiguities in such a way that eq. (1.3) holds. In this sense one is giving a procedure for treating a new class of theories: the non-renormalizable theories obtained from a renormalizable one after the reparameterization of eq. (1.2) [8].

The proof given in [13] for the Weak Equivalence Theorem and in [12] for the Strong Equivalence Theorem heavily relies on the properties of the BPHZ subtraction procedure. It shows that if the T-products are defined according to the BPHZ prescription with some suitably chosen subtraction indices, then eq. (1.3) for the Strong Equivalence Theorem (and respectively eq. (1.4) for the Weak Equivalence Theorem) holds true order by order in the perturbative expansion in ρ. The issue of the validity of these results within different definitions of the T-products was not addressed.

Several algebraic aspects of the Equivalence Theorem in the BRST framework have been discussed in [22]–[27].

Recently the Weak Equivalence Theorem has been investigated by using the BRST techniques in various ways, first in [10] and then in a different context in [11].

In ref. [11] the Equivalence Theorem is shown to hold true provided that both the transformation in eq. (1.2) and its inverse $\varphi = \varphi(\Phi;\rho)$ are local (at least perturbatively in the expansion in ρ).

In our work we prefer to perform a perturbative expansion in term of $\hbar$, which gives also the loop expansion. In particular, the ρ expansion is not equivalent to the loop expansion in the class of transformations

$$\Phi = A_\rho\varphi + \rho\Delta(\varphi)\,, \tag{1.9}$$

where A_ρ is a ρ-dependent linear operator with non-local inverse (e.g. $A_\rho = 1 + \rho_0\Box_x$) and

$\Delta(\varphi) = \{\Delta_i(\varphi)\}$ accounts for the non linear part and is assumed to be a set of Lorentz-invariant monomials in φ and its derivatives. In this case the bilinear part of the action is substantially modified and thus the propagators are ρ-dependent in such a way that the loop expansion differs from the ρ expansion.

This in turn extends the validity of the Equivalence Theorem to a much wider class of field redefinitions, including those in eq. (1.9).

Both in the strong and in the weak formulation, the classical invariance under the BRST transformations is translated into ST identities for the vertex functional. Since ST identities are the ideal tools in order to study the quantum extensions of these symmetries, it is crucial to prove that they are anomaly free. In the present paper we prove that indeed the ST identities can be restored at every order in the loop expansion by using the techniques of removing from the cohomology the terms depending on the ST doublets.

It is interesting to see how this can happen even in the case where the theory is a gauge theory affected by an anomaly in the ST identities associated to the gauge BRST invariance. In this rather peculiar case however the field theory remains sick since unitarity cannot be recovered [28].

Part of the material presented here has been discussed by various authors in the algebraic approach to the Equivalence Theorem. The present paper contributes to focus some important aspects of the ET, in particular the necessity of an extension of the QAP, the cohomology of the linearized classical ST operator (anomalies), the rôle of the locality in the transformation of the field (and its inverse) and finally the use of the Batalin-Vilkovisky in the case of the on-shell treatment of the ET.

The paper is organized as follows. In sections 2 we discuss the Weak Equivalence Theorem. We derive the ST identities and we study their quantum extension. In sections 3 we discuss the Strong Equivalence Theorem by using the path-integral approach. In sections 4 we consider the off-shell formalism and construct a suitable BRST symmetry under which the new action is invariant. In sections 5 the quantum deformation of the ST identities associated to the off-shell case are discussed. In sections 6 we give an on-shell formulation of the Strong Equivalence Theorem and we study the quantum extension of the Slavnov-Taylor identities. In sections 7 we show that the ρ-independence of a selected set of Green functions (to be identified in some cases with the physical content of the theory) is a consequence of the quantum ST identity. In sections 8 we give some applications of the Equivalence Theorem. Conclusions are presented in sections 9. In appendix A we discuss the Quasi Classical Action Principle, and in appendix B we study the removal of the ST doublets from the cohomology.

2. The Weak Equivalence Theorem

In this section we consider the Weak Equivalence Theorem as formulated in eq. (1.4). At classical level we consider the vertex functional

$$\Gamma^{(0)}[\Phi,\beta] = S[\Phi] + \int d^4x\, \beta_i \mathcal{O}_i(\Phi)\,. \tag{2.1}$$

In the spirit of the WET, defined by eq. (1.4), we perform a change of variable, of the kind given in eq. (1.2), inside the classical vertex functional [12]. We write the transformation of the field as

$$\Phi = A_\rho \varphi + \sum_j \rho_j \Delta_j(\varphi)\,, \tag{2.2}$$

where A_ρ is a differential operator reducing to the identity for $\rho = 0$, and $\Delta_j(\varphi)$ are monomials in φ and its derivatives of order at least two in φ.

By performing the field redefinition of eq. (2.2) in $\Gamma^{(0)}$ we obtain the new vertex functional $\tilde{\Gamma}^{(0)}$

$$\tilde{\Gamma}^{(0)}[\varphi, \beta; \rho] = S[\Phi(\varphi; \rho)] + \int d^4x\, \beta_i \mathcal{O}_i(\Phi(\varphi; \rho))\,. \tag{2.3}$$

It should be stressed that the new action is not obtained by a change of variables in the path-integral. Therefore one should not expect the generating functionals to coincide. However there is a very intriguing way, based on the BRST formalism, to formulate the (weak) equivalence of the transformed theory to the original one [9, 11].

The classical vertex functional $\tilde{\Gamma}^{(0)}$ depends on the field φ only through the combination given by $\Phi(\varphi; \rho)$. In order to formulate the WET in terms of BRST invariance we introduce a set of transformations with the requirements that $\Phi(\varphi; \rho)$ is left invariant. This is what we should expect, since $\Phi(\varphi; \rho)$ is the field combination that appears in eq. (1.4). The BRST transformation of φ is dictated by the requirement that $s\Phi(\varphi; \rho) = 0$:

$$0 = s\Phi(x) \; = \; \int d^4y \left(\frac{\delta \Phi(x)}{\delta \varphi(y)}\; s\varphi(y) + \frac{\delta \Phi(x)}{\delta \rho_k(y)}\; s\rho_k(y) \right) \tag{2.4}$$

$$\Rightarrow s\varphi(x) = -\int d^4\, y \left(\frac{\delta \Phi}{\delta \varphi} \right)^{-1}\!\!(x, y) \;\left(\int d^4 z\, \frac{\delta \Phi(y)}{\delta \rho_k(z)}\; s\rho_k(z) \right)\,. \tag{2.5}$$

We introduce a new set of anticommuting external sources $\theta = \{\theta_k\}$ and we take $s\rho_k = \theta_k$, $s\theta_k = 0$. By construction $\tilde{\Gamma}^{(0)}[\varphi, \beta; \rho]$ is invariant under the BRST transformations. Therefore it obeys the following classical ST identity (STI)

$$\int d^4x \left(\frac{\delta \tilde{\Gamma}^{(0)}}{\delta \rho_k(x)}\; \theta_k(x) + \frac{\delta \tilde{\Gamma}^{(0)}}{\delta \varphi(x)}\; s\varphi(x) \right) = 0 \tag{2.6}$$

where (in compact notations)

$$s\varphi = -\left(\frac{\delta \Phi}{\delta \varphi} \right)^{-1} \frac{\delta \Phi}{\delta \rho_k}\; \theta_k\,, \qquad s\rho_k = \theta_k\,, \qquad s\theta_k = 0\,. \tag{2.7}$$

A straightforward computation shows that $s^2 = 0$. The Faddeev-Popov (FP) charge is assigned as follows:

$$\mathrm{FP}(\varphi) = \mathrm{FP}(\rho_k) = 0\,, \qquad \mathrm{FP}(\theta_k) = 1\,. \tag{2.8}$$

The requirement that the vertex functional $\tilde{\Gamma}^{(0)}$ is FP neutral enforces the assignment of the FP charge to the remaining external sources.

Notice that if the operator $\delta\Phi/\delta\varphi$ has no *local* inverse (at least as a formal power series) the present formulation looses most of the nice properties related to BRST invariance. We stress that, in order to extend eq. (2.6) at quantum level, we need to couple $s\varphi$ to its antifield φ^*. This is possible only if $s\varphi$ is local as a formal power series.

2.1 Quantum formulation of the WET

In the sequel we shall discuss the quantum extension of eq. (2.6). In particular we shall show that the classical STI in eq. (2.6) can be fulfilled at the quantum level under the assumption that $\Phi(\varphi;\rho)$ is locally invertible as a formal power series in the fields and external sources and their derivatives.

Then it is straightforward to prove that the quantum extension of eq. (2.6) gives the independence of ρ of any Green function which depends only on $\Phi(\varphi;\rho)$, i.e. one obtains the statement of the WET provided by eq. (1.4).

The WET allows to treat a set of theories, related by a change of fields, in which the jacobian of the transformation does not play any rôle: this is a non-trivial feature of the locally invertible case, already put in evidence in the literature [12, 13]. This feature cannot be properly understood by path-integral arguments. This issue is here analyzed by using BRST techniques.

Once the BRST transformations of the fields φ and the parameters ρ and θ of eq. (2.7) are introduced, one needs to add to the vertex functional a term containing the antifield φ^* coupled to $s\varphi$

$$\tilde{\Gamma}^{(0)}[\varphi,\theta,\varphi^*,\beta;\rho] = S[\Phi(\varphi;\rho)] + \int d^4x\, \varphi^*\ s\varphi + \int d^4x \beta_i \mathcal{O}_i(\Phi(\varphi;\rho))\,. \tag{2.9}$$

This is possible since $s\varphi$ is local as a formal power series in the fields and external sources and their derivatives.

The vertex functional $\tilde{\Gamma}^{(0)}$ satisfies the following Equivalence Theorem STI

$$\mathcal{S}\left(\tilde{\Gamma}^{(0)}\right) \equiv \int d^4x \left(\frac{\delta \tilde{\Gamma}^{(0)}}{\delta \varphi^*} \frac{\delta \tilde{\Gamma}^{(0)}}{\delta \varphi} + \theta_k \frac{\delta \tilde{\Gamma}^{(0)}}{\delta \rho_k} \right) = 0\,. \tag{2.10}$$

Eqs. (2.9)–(2.10) are the starting point for the quantization of the model.

2.2 Quantum restoration of the STI

The first task is then to show that the ET STI in eq. (2.10) can be preserved at the quantum level, i.e. we can define a quantum effective action $\tilde{\Gamma}$ by a suitable choice of non-invariant local counterterms, order by order in the loop expansion, such that

$$\mathcal{S}\left(\tilde{\Gamma}\right) = \int d^4x \left(\frac{\delta \tilde{\Gamma}}{\delta \varphi^*} \frac{\delta \tilde{\Gamma}}{\delta \varphi} + \theta_k \frac{\delta \tilde{\Gamma}}{\delta \rho_k} \right) = 0\,. \tag{2.11}$$

The above STI has the same form as the well-known STI associated with the usual BRST symmetry, arising in ordinary gauge theories. Therefore one expects to be able to characterize the n-th order ST breaking terms by algebraic constraints like the Wess-Zumino consistency condition, under the recursive assumption that the STI have been restored up to order $n-1$.

However the problem is much more intricate here, because almost surely the transformed theory, whose vertex functional is given by $\tilde{\Gamma}$, is not power-counting renormalizable.

Therefore the QAP cannot be applied here to guarantee that the n-th order ST breaking term is a local polynomial in the fields and the external sources and their derivatives.

We wish to comment on this point further. For power-counting renormalizable theories the QAP characterizes the possible breaking of the STI, to all orders in perturbation theory, as the insertion of a suitable local operator with bounded dimension. Consequently, if such an insertion were zero up to order $n-1$, at the n-th order it must reduce to a local polynomial in the fields and the external sources and their derivatives with bounded dimension. This follows from the locality of the insertion and the topological interpretation of the $\hbar$-expansion as a loop-wise expansion.

We stress that all what is needed to carry out the recursive analysis of the ST breaking terms by using algebraic methods is that part of the QAP, saying that at the first non-vanishing order in the loop expansion the ST breaking term is a local polynomial in the fields and the external sources and their derivatives. Again, we wish to emphasize that indeed the QAP says much more, since it characterizes the ST breaking terms to all orders in perturbation theory as the insertion of a suitable local operator. This is true independently of the normalization conditions and of the non-invariant finite action-like counterterms chosen. It might eventually happen that a careful order by order choice of the latter allows to set such a local insertion equal to zero. In this case we speak of a non-anomalous theory.

In the case of a non power-counting renormalizable theory, an all-order extension of the QAP could hardly be proven. This is because several restrictive assumptions on the form of the propagators and on the interaction vertices are required in all known proofs of the QAP. None of them can be imposed with enough generality to non power-counting renormalizable theories.

The key observation [16, 29] is that the full power of the QAP is never used in discussing the recursive restoration of the STI. One only needs the locality of the breaking terms at the first non-vanishing order in the loop expansion. It has then been proposed [16] that a suitable extension of this statement of the QAP might be the good property to look for when dealing with non power-counting renormalizable theories. Such an extension has been called by Stora the Quasi-Classical Action Principle. It states that

> *In the loop-wise perturbative expansion the first non-vanishing order of ST-like identities is a classical local integrated formal power series in the fields and external sources and their derivatives.*

Let us comment on this proposal. Although its plausibility, no satisfactory proof is available for the QCAP. Thus from now on we will assume that it is fulfilled by the regularization used to construct the vertex functional Γ. Moreover, we point out that we do not require that the breaking term is polynomial: loosing the power-counting entails that no bounds on the dimension can in general be expected.

Since we assume the QCAP, we can now use the power of cohomological algebra to constrain the possible ET ST breaking terms and show that the ET STI can always be restored by a suitable order by order choice of non-invariant counterterms.

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The proof is a recursive one. Assume that the ET STI in eq. (2.11) has been fulfilled up to order $n-1$:

$$\mathcal{S}\left(\tilde{\Gamma}\right)^{(j)} = 0\,, \qquad j = 0, \ldots, n-1\,. \tag{2.12}$$

At the next order the ET STI might be broken by the functional $\Delta^{(n)}$ defined by

$$\Delta^{(n)} \equiv \mathcal{S}\left(\tilde{\Gamma}\right)^{(n)}\,. \tag{2.13}$$

Since we are assuming that the QCAP holds, we know that $\Delta^{(n)}$ is a local formal power series in the fields and external sources and their derivatives and has FP-charge +1. No bounds on the dimensions can however be given, since in general we are dealing with a theory which is non-renormalizable by power-counting. This in turn rules out the possibility to constrain $\Delta^{(n)}$ by means of power-counting arguments.

We then must resort to purely algebraic arguments in order to show that $\Delta^{(n)}$ can indeed be removed by a suitable choice of n-th order local non-invariant counterterms. This is a viable way to characterize $\Delta^{(n)}$ because of the following Wess-Zumino consistency condition obeyed by $\Delta^{(n)}$:

$$\mathcal{S}_0\left(\Delta^{(n)}\right) = 0\,, \tag{2.14}$$

where $\mathcal{S}_0$ denotes the classical linearized ST operator

$$\mathcal{S}_0 = \int d^4x \left(\frac{\delta\tilde{\Gamma}^{(0)}}{\delta\varphi^*}\frac{\delta}{\delta\varphi} + \frac{\delta\tilde{\Gamma}^{(0)}}{\delta\varphi}\frac{\delta}{\delta\varphi^*} + \theta_k\frac{\delta}{\delta\rho_k}\right)\,. \tag{2.15}$$

It can be easily proven that $\mathcal{S}_0$ is nilpotent, as a consequence of the classical STI in eq. (2.10). Thus a cohomological analysis of the functional $\Delta^{(n)}$, starting from eq. (2.14), is at hand.

We first notice that the sources (ρ_k, θ_k) form a set of doublets under $\mathcal{S}_0$. In a situation where the counting operator

$$\mathcal{N} = \sum_k \int d^4x \left(\rho_k\frac{\delta}{\delta\rho_k} + \theta_k\frac{\delta}{\delta\theta_k}\right) \tag{2.16}$$

does commute with $\mathcal{S}_0$, the doublets can be removed from the cohomology by standard techniques [30]. Here however the doublets are coupled since $[\mathcal{N}, \mathcal{S}_0] \neq 0$. Under some conditions the coupled doublets can be also removed [31]. Here in particular $\Delta^{(n)}$ in eq. (2.13) is $\mathcal{S}_0$-invariant by the Wess-Zumino consistency condition in eq. (2.14). Moreover since there are neither fields nor external sources with FP-charge +1 apart from θ_k, $\Delta^{(n)}$ must vanish at $\theta_k = 0$. This yields

$$\mathcal{S}_0\left(\left.\Delta^{(n)}\right|_{\theta_k=\rho_k=0}\right) = 0\,. \tag{2.17}$$

In [31] it is shown that the above equation, combined with eq. (2.14), is a sufficient condition to conclude that the functional $\Delta^{(n)} - \left.\Delta^{(n)}\right|_{\theta_k=\rho_k=0}$ is cohomologically trivial, i.e. there exists a local formal power series $\Xi^{(n)}$ such that

$$\Delta^{(n)} - \left.\Delta^{(n)}\right|_{\theta_k=\rho_k=0} = \mathcal{S}_0\left(\Xi^{(n)}\right)\,. \tag{2.18}$$

Moreover, since $\Delta^{(n)}\big|_{\theta_k=\rho_k=0} = 0$, we get from eq. (2.18) that the whole $\Delta^{(n)}$ is $\mathcal{S}_0$-invariant.

By adding to the n-th order effective action $\tilde{\Gamma}^{(n)}$ the n-th order non-invariant counterterm $-\Xi^{(n)}$ we conclude from eq. (2.18) that the STI can be restored at the n-th order, i.e.

$$\mathcal{S}\left(\tilde{\Gamma}\right)^{(n)} = 0\,. \tag{2.19}$$

This ends the proof that the ET STI in eq. (2.11) can be fulfilled to all orders in the loop expansion:

$$\mathcal{S}\left(\tilde{\Gamma}\right) = \int d^4x \left(\frac{\delta\tilde{\Gamma}}{\delta\varphi^*}\frac{\delta\tilde{\Gamma}}{\delta\varphi} + \theta_k\frac{\delta\tilde{\Gamma}}{\delta\rho_k}\right). \tag{2.20}$$

3. The Strong Equivalence Theorem: a path-integral formulation

The main feature of the Strong Equivalence Theorem is that it is obtained by performing a change of variables in the path integral, as already pointed out in eqs. (1.7) and (1.8) .

As in the WET case, the content of eq. (1.3) is here discussed in the framework of BRST theory. However this problem has not straightforward solution.

We find more convenient to introduce a set of auxiliary fields, which allows a linearization of the ST identities. We call it the *off-shell* formalism. We will prove that the related STI can be fulfilled at the quantum level. From the quantum STI, we will deduce the statement of the SET in the off-shell formalism.

We shall also provide the STI relevant for the *on-shell* formulation of the equivalence between the theories given by the generating functionals in eqs. (1.7) and (1.8). The corresponding quantum STI will allow us to prove the validity of the statement of the SET defined by eq. (1.3).

In this section we study the Strong Equivalence Theorem from the point of view of the path-integral formalism. In this framework the field redefinition in eq. (1.2) can be naturally interpreted as a change of variables. This turns out to be very useful in order to derive at the formal level the ST identities controlling the renormalization of the transformed theory.

In the path-integral formalism, one starts from the classical action $S[\Phi]$ and defines a generating functional

$$Z[J,\beta] = \int [\mathcal{D}\Phi]\,\exp\left(iS[\Phi] + i\int d^4x\,J\Phi + i\int d^4x\,\beta_i\mathcal{O}_i(\Phi)\right)\,. \tag{3.1}$$

In the above equation we have introduced the external sources β_i, coupled to the local operators $\mathcal{O}_i(\Phi)$.

We now perform the field redefinition

$$\Phi = A_\rho\varphi + \sum_j \rho_j\Delta_j(\varphi)\,. \tag{3.2}$$

Performing the change of variables of eq. (3.2) in eq. (3.1) one gets

$$\begin{aligned}\tilde{Z}[J,\beta] &= \int[\mathcal{D}\varphi]\det\left(\frac{\delta\Phi}{\delta\varphi}\right)\exp\left(iS[\Phi(\varphi;\rho)] + i\int d^4xJ\Phi(\varphi;\rho) + i\int d^4x\beta_i\mathcal{O}_i(\Phi(\varphi;\rho))\right)\\ &= \lim_{K,\eta,\bar{\eta}\to 0}\tilde{Z}_{on}[J,\beta,K,\eta,\bar{\eta};\rho]\end{aligned} \tag{3.3}$$

where we have introduced:

$$\tilde{Z}_{on}[J,\beta,K,\eta,\bar{\eta};\rho] = \int [\mathcal{D}\varphi][\mathcal{D}\bar{c}][\mathcal{D}c]\exp\Bigg(iS[\Phi(\varphi;\rho)] + i\int d^4x\bar{c}\left(\frac{\delta\Phi}{\delta\varphi}c + \frac{\delta\Phi}{\delta\rho_k}\theta_k\right) + $$
$$+ i\int d^4xJ\Phi(\varphi;\rho) + i\int d^4x\beta_i\mathcal{O}_i(\chi) +$$
$$+ i\int d^4xK\varphi + i\int d^4x\eta\bar{c} + i\int d^4x\bar{\eta}c\Bigg). \quad (3.4)$$

To deal with the jacobian of the transformation, we use the standard trick of introducing the Faddeev-Popov ghost and antighost $c,\bar{c}$ to exponentiate the determinant. For later convenience we have also added to the action the term

$$i\int d^4x\,\bar{c}\frac{\delta\Phi}{\delta\rho_k}\theta_k\,.$$

The external sources θ_k have opposite statistics with respect to ρ_k and will be interpreted as before as the BRST partners of ρ_k. In eq. (3.4) we need the sources K, η and $\bar{\eta}$ coupled to the quantum fields φ, $\bar{c}$ and c, in order to define the perturbative expansion of the theory.

The external source J is now associated to the *composite* (from the point of view of φ) operator $\Phi(\varphi;\rho)$. It is on the same footing as the external sources β. The limit $K,\eta,\bar{\eta}\to 0$ means that in the end we care about amplitudes with Φ or $\mathcal{O}(\Phi)$ external legs.

We will prove in sections 6, by using the method of Batalin-Vilkovisky, that the construction of the Green functions can be equivalently performed via the generating functional Z or $\tilde{Z}_{on}$. This is the on-shell version of the SET.

A less involved formulation, from an algebraic point of view, is obtained by introducing a couple of auxiliary fields B and χ. That is, we rewrite $Z[J,\beta]$ as

$$\tilde{Z}[J,\beta] = \int [\mathcal{D}\varphi][\mathcal{D}\chi]\delta(\chi-\Phi(\varphi;\rho))\det\left(\frac{\delta\Phi}{\delta\varphi}\right)\exp\left(iS[\chi] + i\int d^4xJ\chi + i\int d^4x\beta_i\mathcal{O}_i(\chi)\right)$$
$$= \lim_{K,J_B,\eta,\bar{\eta}\to 0}\tilde{Z}_{\rm off}[J,\beta,K,J_B,\eta,\bar{\eta};\rho] \quad (3.5)$$

where now

$$\tilde{Z}_{\rm off}[J,\beta,K,J_B,\eta,\bar{\eta};\rho] = \int [\mathcal{D}\varphi][\mathcal{D}\chi][\mathcal{D}B][\mathcal{D}\bar{c}][\mathcal{D}c]\times$$
$$\times\exp\Bigg(iS[\chi] + i\int d^4xB(\chi-\Phi(\varphi;\rho)) +$$
$$+ i\int d^4x\bar{c}\left(\frac{\delta\Phi}{\delta\varphi}c + \frac{\delta\Phi}{\delta\rho_k}\theta_k\right) + i\int d^4xJ\chi +$$
$$+ i\int d^4x\eta\bar{c} + i\int d^4x\bar{\eta}c + i\int d^4xK\varphi +$$
$$+ i\int d^4xJ_BB + i\int d^4x\beta_i\mathcal{O}_i(\chi)\Bigg). \quad (3.6)$$

We speak in this case of the off-shell formulation of the Equivalence Theorem. In eq. (3.6) K and J_B are respectively the sources coupled to φ and B. Notice that J is now coupled to χ.

As much as in the on-shell case, there exists a ST identity expressing the ρ-independence of the Green functions involving solely χ and $\mathcal{O}_i(\chi)$. Moreover, the ST identity is linear in the off-shell case. Despite of this advantage, the calculations in the off-shell scheme are clumsy due to the high number of quantized fields involved. We will illustrate the derivation of the ST identities in the off-shell case in sections 4.

The on-shell and off-shell cases can be related by eliminating the auxiliary fields. This will be discussed in sections 5.

4. The off-shell fomulation of the SET

4.1 The classical Slavnov-Taylor identities

We discuss here the construction of the BRST transformations and related ST identities in the off-shell case. For that purpose we need to work on the Legendre transform $\tilde{\Gamma}_{\rm off}$ of the connected generating functional $\tilde{W}_{\rm off}$, where we have defined

$$\tilde{Z}_{\rm off} = \exp\left(\frac{i}{\hbar}\tilde{W}_{\rm off}\right). \tag{4.1}$$

From eq. (3.6) we deduce the classical vertex functional $\tilde{\Gamma}^{(0)}_{\rm off}$

$$\tilde{\Gamma}^{(0)}_{\rm off} \equiv S[\chi] + \int d^4x B(\chi - \Phi(\varphi;\rho)) + \int d^4x \bar{c}\left(\frac{\delta\Phi}{\delta\varphi}c + \frac{\delta\Phi}{\delta\rho_k}\theta_k\right) + \int d^4x \beta_i \mathcal{O}_i(\chi)\,. \tag{4.2}$$

The BRST differential can be easily written by looking at the classical vertex functional $\tilde{\Gamma}^{(0)}_{\rm off}$. Let us define

$$\begin{array}{llll} s\varphi = c, & s\bar{c} = B, & s\rho_k = \theta_k, & s\chi = 0, \\ sc = 0, & sB = 0, & s\theta_k = 70\,. & \end{array} \tag{4.3}$$

We assign the corresponding FP-charge by setting

$$\mathrm{FP}(\varphi) = \mathrm{FP}(B) = \mathrm{FP}(\rho_k) = \mathrm{FP}(\chi) = 0\,, \qquad \mathrm{FP}(c) = \mathrm{FP}(\theta_k) = 1\,, \qquad \mathrm{FP}(\bar{c}) = -1\,. \tag{4.4}$$

The FP-charge of the external sources can be read off by imposing the requirement that the classical action $\tilde{\Gamma}^{(0)}_{\rm off}$ is FP-neutral. The vertex functional $\tilde{\Gamma}^{(0)}_{\rm off}$ in eq. (4.2) can then be cast in the form

$$\tilde{\Gamma}^{(0)}_{\rm off} = S[\chi] + s\int d^4x \bar{c}\,(\chi - \Phi(\varphi;\rho)) + \int d^4x \beta_i \mathcal{O}_i(\chi). \tag{4.5}$$

With this procedure one has automatically $sS[\chi] = 0$ and $s\mathcal{O}_i(\chi) = 0$. We understand that the operators $\mathcal{O}_i(\Phi)$ now become the operators $\mathcal{O}_i(\chi)$. $\tilde{\Gamma}^{(0)}_{\rm off}$ is s-invariant. A similar approach has been adopted in [25] where the issue of the cohomology is not dealt with.

Notice that in this formulation the whole Φ dependence appears through the cohomologically trivial term

$$\int d^4x B(\chi - \Phi(\varphi;\rho)) + \int d^4x \bar{c}\left(\frac{\delta\Phi}{\delta\varphi}c + \frac{\delta\Phi}{\delta\rho_k}\theta_k\right) = s\int d^4x \bar{c}\,(\chi - \Phi(\varphi;\rho))\,.$$

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The change of variable in the path integral has given rise to an ET ST identity of the transformed action:

$$\mathcal{S}\left(\tilde{\Gamma}^{(0)}_{\rm off}\right) = \int d^4x \, \left(c \frac{\delta \tilde{\Gamma}^{(0)}_{\rm off}}{\delta \varphi} + B \frac{\delta \tilde{\Gamma}^{(0)}_{\rm off}}{\delta \bar{c}} + \theta_k \frac{\delta \tilde{\Gamma}^{(0)}_{\rm off}}{\delta \rho_k} \right) = 0 \, . \tag{4.6}$$

This ET STI is linear, hence it does not require the introduction of any antifield. Moreover, it can be seen from eq. (4.6) that all fields and external sources transforming under $\mathcal{S}$ form a set of (decoupled) doublets ([31] and appendix B), given by (φ, c), $(\bar{c}, B)$ and (ρ_k, θ_k). Therefore the study of the cohomology of $\mathcal{S}$ is particularly simple. This is a significant advantage allowing to treat also those situations where $\Phi = \Phi(\varphi;\rho)$ does not have a local inverse, by using algebraic techniques.

5. Quantization of the ET STI in the off-shell formalism

We now study the quantization of the ET STI in the off-shell case. Things are much simpler here due to the fact that all fields and external sources entering in the ET STI form sets of ordinary[1] doublets.

The proof that the classical ET STI in eq. (4.6) can be extended to all orders in the loop expansion is as follows. We assume that the ET STI have been restored up to order $n-1$, i.e.

$$\mathcal{S}(\tilde{\Gamma}_{\rm off})^{(j)} = \int d^4x \, \left(c \frac{\delta \tilde{\Gamma}^{(j)}_{\rm off}}{\delta \varphi} + B \frac{\delta \tilde{\Gamma}^{(j)}_{\rm off}}{\delta \bar{c}} + \theta_k \frac{\delta \tilde{\Gamma}^{(j)}_{\rm off}}{\delta \rho_k} \right) = 0 \, , \qquad j = 0, 1, \ldots, n-1 \, .$$

Thanks to the QCAP at the n-th order the possible ST breaking terms

$$\Delta^{(n)} \equiv \mathcal{S}(\tilde{\Gamma}_{\rm off})^{(n)} = \mathcal{S}(\tilde{\Gamma}^{(n)}_{\rm off}) = \int d^4x \, \left(c \frac{\delta \tilde{\Gamma}^{(n)}_{\rm off}}{\delta \varphi} + B \frac{\delta \tilde{\Gamma}^{(n)}_{\rm off}}{\delta \bar{c}} + \theta_k \frac{\delta \tilde{\Gamma}^{(n)}_{\rm off}}{\delta \rho_k} \right)$$

are local integrated formal power series in the fields, the external sources and their derivatives. Notice that $\mathcal{S}$ is linear in the quantum fields. Therefore no antifields are needed in the off-shell formalism. This in turn yields that $\Delta^{(n)}$ only depends on the n-th order vertex functional $\tilde{\Gamma}^{(n)}_{\rm off}$ and does not receive contributions from lower order terms, unlike in the on-shell case.

The Wess-Zumino consistency condition reads:

$$\mathcal{S}\Delta^{(n)} = \int d^4x \, \left(c \frac{\delta \Delta^{(n)}}{\delta \varphi} + B \frac{\delta \Delta^{(n)}}{\delta \bar{c}} + \theta_k \frac{\delta \Delta^{(n)}}{\delta \rho_k} \right) = 0 \, . \tag{5.1}$$

We observe that the counting operator for the $\mathcal{S}$-doublets (φ, c), $(\bar{c}, B)$ and (ρ_k, θ_k) commutes with $\mathcal{S}$. Therefore we can introduce a homotopy operator $\mathcal{K}$ [30] trivializing the cohomology of $\mathcal{S}$. $\mathcal{K}$ is defined as

$$\mathcal{K} = \int_0^1 dt \int d^4x \, \left(\varphi \lambda_t \frac{\delta}{\delta c} + \bar{c} \lambda_t \frac{\delta}{\delta B} + \rho_k \lambda_t \frac{\delta}{\delta \theta_k} \right) , \tag{5.2}$$

[1] Decoupled, according to the terminology of [31].

where λ_t acts as follows on an arbitrary functional X depending on $\varphi, c, \bar{c}, B, \rho_k, \theta_k$ and on other variables ζ:

$$\lambda_t X[\varphi, c, \bar{c}, B, \rho_k, \theta_k, \zeta] = X[t\varphi, tc, t\bar{c}, tB, t\rho_k, t\theta_k, \zeta]\,. \tag{5.3}$$

By explicit computation one verifies that

$$\{\mathcal{K}, \mathcal{S}\} X = X\,, \tag{5.4}$$

where in the above equation $X[\varphi, c, \bar{c}, B, \rho_k, \theta_k, \zeta]$ is assumed to vanish at $c = \theta_k = 0$ (as it happens for $\Delta^{(n)}$, since it has FP-charge $+1$ and the only fields and external sources with FP-charge $+1$ are c and θ_k). We apply eq. (5.4) to $\Delta^{(n)}$ and use the Wess-Zumino consistency condition in eq. (5.1). We arrive at

$$\Delta^{(n)} = \{\mathcal{K}, \mathcal{S}\}\Delta^{(n)} = \mathcal{S}\mathcal{K}\Delta^{(n)}\,. \tag{5.5}$$

Therefore $-\Xi^{(n)} = \mathcal{K}\Delta^{(n)}$ are the counterterms to be added to the n-th order quantum effective action $\tilde{\Gamma}^{(n)}_{\rm off}$ to guarantee that the STI are fulfilled at the n-th order in the loop expansion. Notice that by the QCAP $\Xi^{(n)}$ is local in the sense of integrated formal power series.

We have proven that the ET STI can be fulfilled in the off-shell formalism, i.e. it is possible to define to all orders in the loop expansion a quantum effective action $\tilde{\Gamma}_{\rm off}$ such that

$$\mathcal{S}(\tilde{\Gamma}_{\rm off}) = \int d^4x \left(c\frac{\delta\tilde{\Gamma}_{\rm off}}{\delta\varphi} + B\frac{\delta\tilde{\Gamma}_{\rm off}}{\delta\bar{c}} + \theta_k\frac{\delta\tilde{\Gamma}_{\rm off}}{\delta\rho_k}\right) = 0\,. \tag{5.6}$$

The above equation ensures that the Green functions of local (in the sense of formal power series) BRST invariant operators are ρ-independent. The proof is reported in sections 7.

Notice that every counter term of the form $\mathcal{M} = \mathcal{M}(\chi)$ is such that

$$\mathcal{S}(\mathcal{M}) = \int d^4x \left(c\frac{\delta\mathcal{M}}{\delta\varphi} + B\frac{\delta\mathcal{M}}{\delta\bar{c}} + \theta_k\frac{\delta\mathcal{M}}{\delta\rho_k}\right) = 0\,;$$

that is it can be safely added to the n-th order ST invariant normalization conditions without altering the ST identity. Thus the physical conditions imposed on the theory for $\rho = 0$ can be preserved in the theory with $\rho \neq 0$.

6. The SET in the on-shell formalism

The Strong Equivalence Theorem can be formulated in the framework of BRST invariance also without introducing auxilary fields (on-shell formalism). However one needs to use the formalism of Batalin-Vilkovisky to full extent. The starting point is the following on-shell classical action (for convenience we leave out the $\tilde{}$ and the subscript $_{on}$ from Γ):

$$\Gamma^{(0)}_0 = S[\Phi(\varphi;\rho), \beta] + \int d^4x \bar{c} s \Phi(\varphi;\rho)\,, \tag{6.1}$$

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where the superscript ${}^{(0)}$ denotes the zero-th order in the loop expansion (the classical approximation), and the subscript ${}_0$ refers to the order in the antighost $\bar{c}^*$ (conjugated to the antighost field $\bar{c}$) to be shortly introduced. The full on-shell classical action $\Gamma^{(0)}$ will eventually include terms of all orders in $\bar{c}^*$.

S depends on the field Φ and on a set of external sources β coupled to local composite operators $\mathcal{O}(\Phi)$, functions of Φ only.

The BRST differential s is defined as

$$s\varphi = c\,, \qquad sc = 0\,, \qquad s\rho = \theta\,, \qquad s\theta = 0\,. \tag{6.2}$$

That is, (φ, c) and (ρ, θ) enter as doublets under s. In order to achieve the BRST invariance of $\Gamma_0^{(0)}$ we need to define the BRST variation of $\bar{c}$ as follows:

$$s\bar{c} = -\frac{\delta S}{\delta \Phi}\,. \tag{6.3}$$

Since $s^2\Phi(\varphi;\rho) = 0$ the above equation guarantees that $\Gamma_0^{(0)}$ is BRST invariant.

We remark that the BRST differential s is nilpotent only on-shell, since

$$s^2\bar{c} = -\frac{\delta^2 S}{\delta\Phi\delta\Phi}\, s\Phi = -\frac{\delta^2 S}{\delta\Phi\delta\Phi}\frac{\delta\Gamma_0^{(0)}}{\delta\bar{c}}\,. \tag{6.4}$$

As is well known, in order to define the composite operator $s\bar{c}$ at the quantum level we need to couple it in the classical action to the corresponding antifield $\bar{c}^*$. We denote by Σ_1 those terms of the classical action of order one in $\bar{c}^*$:

$$\Sigma_1 = -\int d^4x \bar{c}^* \frac{\delta S}{\delta\Phi}\,. \tag{6.5}$$

The new classical action is

$$\Gamma_1^{(0)} = \Gamma_0^{(0)} + \Sigma_1\,. \tag{6.6}$$

$\Gamma_0^{(0)}$ is BRST invariant, however $\Gamma_1^{(0)}$ is not:

$$\begin{aligned} s\Gamma_1^{(0)} &= -\int d^4x \bar{c}^* \frac{\delta^2 S}{\delta\Phi\delta\Phi} s\Phi \\ &= -\int d^4x \bar{c}^* \frac{\delta^2 S}{\delta\Phi\delta\Phi}\frac{\delta\Gamma_1^{(0)}}{\delta\bar{c}}\,. \end{aligned} \tag{6.7}$$

The breaking term in the r.h.s. of the above equation is linear in $\bar{c}^*$. The BRST invariance of the classical action is only obtained in the on-shell limit $\frac{\delta\Gamma_0^{(0)}}{\delta\bar{c}} = \frac{\delta\Gamma_1^{(0)}}{\delta\bar{c}} = 0$.

This difficulty can be circumvented by making full use of the Batalin-Vilkovisky formalism. First we define the ST operator as

$$\mathcal{S}(X) = \delta X + (X, X) \tag{6.8}$$

where the parenthesis is given by

$$(X, Y) = \int d^4x \frac{\delta X}{\delta\bar{c}^*}\frac{\delta Y}{\delta\bar{c}} \tag{6.9}$$

and the short-hand δ is for the linear part of the ST operator

$$\delta X = \int d^4x \left(c\frac{\delta X}{\delta\varphi} + \theta\frac{\delta X}{\delta\rho} \right) . \tag{6.10}$$

Then, in order to implement at the classical level the ST identities without going on-shell with $\bar{c}$ (i.e. without imposing $\frac{\delta\Gamma^{(0)}}{\delta\bar{c}} = s\Phi = 0$) we add to $\Gamma^{(0)}_0$ monomials of higher degree in $\bar{c}^*$.

Thus we will construct the full on-shell classical action $\Gamma^{(0)}$ by providing all of its $\bar{c}^*$-dependent components Σ_j. Σ_j is assumed to be of order j in $\bar{c}^*$. Moreover the Σ_j fulfill for $j > 0$

$$\frac{\delta\Sigma_j}{\delta\bar{c}} = 0 \tag{6.11}$$

so that the full on-shell classical action satisfies the ghost equation

$$\frac{\delta\Gamma^{(0)}}{\delta\bar{c}} = \frac{\delta\Gamma^{(0)}_0}{\delta\bar{c}} = s\Phi \, . \tag{6.12}$$

The functionals Σ_j are constructed as follows.

Eq. (6.7) is translated into

$$\mathcal{S}(\Gamma^{(0)}_1) = -\int d^4x \bar{c}^* \frac{\delta^2 S}{\delta\Phi\delta\Phi}\frac{\delta\Gamma^{(0)}_1}{\delta\bar{c}} \, . \tag{6.13}$$

We begin by adding the term Σ_2 quadratic in $\bar{c}^*$:

$$\Sigma_2 = \int d^4x \frac{1}{2}\left(\bar{c}^*\right)^2 \frac{\delta^2 S}{\delta\Phi\delta\Phi} \tag{6.14}$$

and define

$$\Gamma^{(0)}_2 = \Sigma_0 + \Sigma_1 + \Sigma_2 \, . \tag{6.15}$$

It turns out that $\mathcal{S}(\Gamma^{(0)}_2)$ is quadratic in $\bar{c}^*$:

$$\begin{aligned} \mathcal{S}(\Gamma^{(0)}_2) &= \mathcal{S}(\Gamma^{(0)}_1) + \delta\Sigma_2 + (\Gamma^{(0)}_1, \Sigma_2) + (\Sigma_2, \Gamma^{(0)}_1) + (\Sigma_2, \Sigma_2) \\ &= -\int d^4x \bar{c}^* \frac{\delta^2 S}{\delta\Phi\delta\Phi}\frac{\delta\Gamma^{(0)}_2}{\delta\bar{c}} + s\Sigma_2 + \int d^4x \bar{c}^* \frac{\delta^2 S}{\delta\Phi\delta\Phi}\frac{\delta\Gamma^{(0)}_2}{\delta\bar{c}} \\ &= s\Sigma_2 = \int d^4x \frac{1}{2}(\bar{c}^*)^2 \frac{\delta^3 S}{\delta\Phi\delta\Phi\delta\Phi} s\Phi \\ &= \int d^4x \frac{1}{2}(\bar{c}^*)^2 \frac{\delta^3 S}{\delta\Phi\delta\Phi\delta\Phi}\frac{\delta\Gamma^{(0)}_2}{\delta\bar{c}} \, . \end{aligned} \tag{6.16}$$

The construction can be iterated. Define for $j > 0$

$$\Sigma_j \equiv (-1)^j \frac{1}{j!} \int d^4x (\bar{c}^*)^j \frac{\delta^{(j)} S}{\delta\Phi \ldots \delta\Phi} \, . \tag{6.17}$$

while for $j = 0$ we set

$$\Sigma_0 = \Gamma^{(0)}_0 \, . \tag{6.18}$$

Then

$$\Gamma^{(0)}_n = \sum_{j=0}^{n} \Sigma_j \tag{6.19}$$

is such that

$$\begin{aligned} \mathcal{S}(\Gamma^{(0)}_n) &= (-1)^n \frac{1}{n!} \int d^4x (\bar{c}^*)^n \frac{\delta^{(n+1)} S}{\delta\Phi \dots \delta\Phi} s\Phi \\ &= (-1)^n \frac{1}{n!} \int d^4x (\bar{c}^*)^n \frac{\delta^{(n+1)} S}{\delta\Phi \dots \delta\Phi} \frac{\delta\Gamma^{(0)}_n}{\delta\bar{c}} . \end{aligned} \tag{6.20}$$

Thus the breaking of the classical ST identity for the functional $\Gamma^{(0)}_n$ is of order n in $\bar{c}^*$. Therefore the full on-shell classical action, defined by

$$\Gamma^{(0)} = \sum_{j=0}^{\infty} \Sigma_j \, , \tag{6.21}$$

fulfills the classical ST identity

$$\mathcal{S}(\Gamma^{(0)}) = 0 \, . \tag{6.22}$$

In particular, if $S[\Phi, \beta]$ is a polynomial of degree M in Φ, the classical action $\Gamma^{(0)}_M$ defined according to eq. (6.19) fulfills the ST identity

$$\mathcal{S}(\Gamma^{(0)}_M) = 0 \, . \tag{6.23}$$

Notice that already at the classical level there are mixed couplings $\bar{c}^* - \beta$, due to the dependence of $\delta S/\delta\Phi$ on β. This does not prevent the identification of β with the source coupled to the operator $\mathcal{O}(\Phi)$, since

$$\left. \frac{\delta\Gamma^{(0)}_M}{\delta\beta} \right|_{\beta=\bar{c}^*=0} = \mathcal{O}(\Phi) \, . \tag{6.24}$$

6.1 Quantization

We study the quantization of the on-shell ET STI. We will show that it can be fulfilled by a suitable choice of local finite counter-terms order by order in the loop expansion [32].

The proof is a recursive one. Let us assume that the ST identity has been restored up to order $n-1$, so that

$$\mathcal{S}(\Gamma)^{(j)} = 0 \, , \qquad j = 0, 1, \dots, n-1 \, . \tag{6.25}$$

The n-th order ST breaking term $\Delta^{(n)}$, defined by

$$\Delta^{(n)} \equiv \mathcal{S}(\Gamma)^{(n)} \, , \tag{6.26}$$

fulfills the following Wess-Zumino consistency condition:

$$\mathcal{S}_0 \Delta^{(n)} = 0 \, , \tag{6.27}$$

where $\mathcal{S}_0$ is the classical linearized ST operator given by

$$\mathcal{S}_0(X) \equiv \int d^4x \left(c\frac{\delta X}{\delta\varphi} + \theta\frac{\delta X}{\delta\rho} \right) + (\Gamma^{(0)}, X) + (X, \Gamma^{(0)}) \,. \tag{6.28}$$

Moreover thanks to the QCAP $\Delta^{(n)}$ is a local integrated formal power series with FP-charge $+1$. Since the only fields and external sources with FP-charge $+1$ are c and θ, we get that

$$\Delta^{(n)}\Big|_{c=\theta=0} = 0 \,. \tag{6.29}$$

Notice that both c and θ enter into doublets under the classical linearized ST operator $\mathcal{S}_0$. However in the present case (φ, c) and (ρ, θ) are coupled doublets, according to the terminology of [31], since their counting operator does not commute with $\mathcal{S}_0$. Therefore we must resort to the results discussed in [31] and summarized in appendix B to carry out the recursive cohomological analysis of the ST breaking term $\Delta^{(n)}$.

By using these results we get that eq. (6.29) combined with eq. (6.27) guarantees that $\Delta^{(n)}$ belongs to the trivial cohomology class of the operator $\mathcal{S}_0$, namely there exists a local integrated formal power series $\Xi^{(n)}$ such that

$$\mathcal{S}_0(-\Xi^{(n)}) = \Delta^{(n)} \,. \tag{6.30}$$

By adding the local (in the sense of formal power series) finite counterterms $\Xi^{(n)}$ to the n-th order effective action we end up with a new symmetric quantum effective action fulfilling the ST identity up to order n.

We conclude that it is possible to define a full quantum effective action Γ fulfilling the ST identity

$$\mathcal{S}(\Gamma) = 0 \,, \tag{6.31}$$

to all orders in the loop expansion. We remark that Γ also depends on the external sources β coupled to local composite operators $\mathcal{O}(\Phi)$ which are functions only of Φ. Notice that, unlike in the standard BRST treatments of gauge theories, the sources β are not coupled here to BRST invariant operators. All the same we will be able to prove the ρ-independence of the vertex functions

$$\frac{\delta^{(n)}\tilde{\Gamma}_{on}}{\delta\beta_{i_1}(x_1)\ldots\delta\beta_{i_n}(x_n)} \tag{6.32}$$

as a consequence of the ET STI in eq. (6.31), once all the external sources are switched off and the conditions $\frac{\delta\tilde{\Gamma}_{on}}{\delta c}$=0, $\frac{\delta\tilde{\Gamma}_{on}}{\delta\bar{c}}$=0 and $\frac{\delta\tilde{\Gamma}_{on}}{\delta\varphi}$=0 are imposed.

7. Consequences for the Green functions of the model

We now have to investigate the consequences of the ET STI on the Green functions of the model. We have identified for each case of the ET a selected set of Green functions:

(a) for the WET those of the BRST invariant operators, including operators dependent only on $\Phi(\varphi;\rho)$ (since in this formalism $s\Phi = 0$);

(*b*) for the off-shell SET those of the BRST invariant operators, including operators dependent only on χ (since in this formalism $s\chi = 0$);

(*c*) for the on-shell SET those of the operators dependent only on $\Phi(\varphi; \rho)$. Notice that in this case these are not BRST invariant. The external source coupled to this operator must be included in the Batalin-Vilkovisky formalism.

In each case the relevant ET STI imply that the above mentioned Green functions of local (in the sense of formal power series) operators do not depend on ρ.

In some cases this set of Green functions can be identified with the physics of the model. However this is not always possible. For instance it can be shown, in [33], that the ET provides a somewhat natural quantization prescription for those gauge theories which are anomalous in the ordinary BRST quantization. Although the ET STI can be restored in these models to all orders in the loop expansion (assuming that the QCAP holds), their content is much weaker than that of ordinary STI based on gauge symmetry. In particular the former, unlike the latter, say nothing about the unitarity of the theory, which is indeed violated at the quantum level. Therefore we cannot use ET STI to study physical observables.

For this reason we limit ourselves to the discussion of the ρ-independence of the relevant Green functions discussed above, with no claim that they identify the physics of the model.

We work out in detail the proof of the independence of ρ for the Green functions of operators dependent only on Φ for the on-shell SET case. The same pattern applies to the other two cases, where the proof boils down to a straightforward paraphrase of standard arguments [29], previously used to discuss the dependence of the BRST invariant operators on the gauge parameters in ordinary gauge theories. Instead in the on-shell case we are dealing with operators which are not BRST invariant. However it turns out that a similar proof can be applied also in this case.

We start from the ST identity for the connected generating functional $\tilde{W}_{on}$

$$\mathcal{S}(\tilde{W}_{on}) \equiv \int d^4x \left(-K(x) \frac{\delta \tilde{W}_{on}}{\delta \bar{\eta}(x)} - \eta(x) \frac{\delta \tilde{W}_{on}}{\delta \bar{c}^*(x)} + \theta_i(x) \frac{\delta \tilde{W}_{on}}{\delta \rho_i(x)} \right) = 0 \,. \tag{7.1}$$

We differentiate the above equation with respect to $\theta_k(y)$ and get

$$\frac{\delta \tilde{W}_{on}}{\delta \rho_k(y)} = \int d^4x \left(K(x) \frac{\delta^2 \tilde{W}_{on}}{\delta \theta_k(y) \delta \bar{\eta}(x)} + \eta(x) \frac{\delta^2 \tilde{W}_{on}}{\delta \theta_k(y) \delta \bar{c}^*(x)} + \theta_i(x) \frac{\delta^2 \tilde{W}_{on}}{\delta \theta_k(y) \delta \rho_i(x)} \right) . \tag{7.2}$$

Differentiation of eq. (7.2) with respect to the sources $\beta_{l_1}(z_1), \ldots, \beta_{l_n}(z_n)$ gives, once we go on shell and set $K = \theta = \beta = \eta = \bar{\eta} = \bar{c}^* = 0$:

$$\left. \frac{\delta^{(n+1)} \tilde{W}_{on}}{\delta \beta_{l_1}(z_1) \ldots \delta \beta_{l_n}(z_n) \delta \rho_k(y)} \right|_{K=\theta=\beta=\eta=\bar{\eta}=\bar{c}^*=0} = 0 \,.$$

Therefore the Green functions of operators depending only on $\Phi(\varphi; \rho)$ are independent of ρ_k. This is the content of the on-shell SET.

8. Example: an application of the ET in the abelian gauge theory

We want to illustrate our discussion by giving an example obtained from the abelian gauge theory. In this section we show that there exists a very simple field redefinition connecting Proca's gauge, which looks unitary but is not renormalizable by power-counting, to the Lorentz-covariant gauge, which is renormalizable by power-counting. This procedure based on ET might give a meaning to the massive QED in Proca's gauge, which is non-renormalizable in the conventional sense.

8.1 The starting point: the Lorentz-covariant gauge

We start with the lagrangian including matter fields with a gauged $U(1)$ symmetry, where the gauge boson is massive and is quantized in the Lorentz-covariant gauge, parameterized by α:

$$S_\alpha = \int d^4x\, \mathcal{L}_\alpha = \int d^4x \left(-\frac{1}{4} F_{\mu\nu}F^{\mu\nu} + \frac{m^2}{2}A^2 - \frac{\alpha}{2}(\partial A)^2 + \mathcal{L}_\psi(A_\mu, \psi, \bar{\psi}) \right). \tag{8.1}$$

$$\mathcal{L}_\psi = \bar{\psi}\not{D}\psi - m\bar{\psi}\psi \tag{8.2}$$

is the term containing the fermion fields, which are minimally coupled to the gauge boson (i.e. $\not{D} = \not{\partial} + ie\not{A}$).

8.2 From the Lorentz-covariant gauge to Proca's gauge: a field redefinition

We can use the Equivalence Theorem at the classical level to connect a theory with mass m and gauge parameter α to a theory with the same mass m and gauge parameter $\alpha = 0$. That is, we are dealing with a field transformation connecting the Lorentz-covariant gauge to Proca's gauge.

The gauge provided by the action in eq. (8.1) can also be obtained by using the auxiliary field φ:

$$\begin{aligned} S &= \int d^4x\, \mathcal{L}(x) \\ &= \int d^4x \left(-\frac{1}{4} F_{\mu\nu}F^{\mu\nu} + \frac{m^2}{2}A^2 + m\sqrt{\alpha}\varphi\partial A + \frac{m^2}{2}\varphi^2 + \mathcal{L}_\psi(A_\mu, \psi, \bar{\psi}) \right). \end{aligned} \tag{8.3}$$

The field φ does not enter in the interaction vertices. Thus we can impose the equation of motion for φ:

$$\frac{\delta S}{\delta \varphi} = m\sqrt{\alpha}\partial A + m^2\varphi = 0 \tag{8.4}$$

so that

$$\varphi = -\frac{\sqrt{\alpha}}{m}\partial A\,. \tag{8.5}$$

Both equations (8.4) and (8.5), being linear in the fields, are valid at the quantum level.

If we perform the following field redefinition in eq. (8.3)

$$\begin{aligned} &A_\nu \to A'_\nu \equiv A_\nu + \frac{\sqrt{\alpha}}{m}\partial_\nu\varphi\,, && \varphi \to \varphi\,, \\ &\psi \to \psi' \equiv \exp\left(ie\frac{\sqrt{\alpha}}{m}\varphi \right)\psi\,, && \bar{\psi} \to \bar{\psi}' \equiv \exp\left(-ie\frac{\sqrt{\alpha}}{m}\varphi \right)\bar{\psi}\,, \end{aligned} \tag{8.6}$$

we obtain the transformed action

$$\begin{aligned} S_{\text{Proca}} &= \int d^4x\, \mathcal{L}_{\text{Proca}}(x) \\ &= \int d^4x \left(-\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \frac{m^2}{2} A_\mu^2 + \mathcal{L}_{\text{int}}(A_\mu, \psi, \bar{\psi}) + \alpha \partial_\nu \varphi \partial^\nu \varphi + \frac{m^2}{2} \varphi^2 \right), \end{aligned} \tag{8.7}$$

which is the QED action in Proca's gauge with the addition of the free scalar field φ.

8.3 Proca's gauge

Notice that in a massive abelian gauge theory with zero gauge-fixing parameter the propagator given by

$$D^{\mu\nu} = -i \frac{g^{\mu\nu} - p^\mu p^\nu / m^2}{p^2 - m^2 + i\epsilon}, \tag{8.8}$$

behaves asymptotically as a constant for large p^2 and propagates only transverse polarizations. In the Lorentz-covariant gauge

$$D_\alpha^{\mu\nu}(p) = -i \frac{g^{\mu\nu} - p^\mu p^\nu / m^2}{p^2 - m^2 + i\epsilon} - i \frac{p^\mu p^\nu / m^2}{p^2 - m^2/\alpha + i\epsilon} \tag{8.9}$$

behaves as $1/p^2$ at large momenta and propagates both spin 1 and (unphysical) spin zero modes. As a consequence, the latter theory can be considered renormalizable by power-counting but not manifestly unitary, while the former is not renormalizable by power-counting but is manifestly unitary.

Therefore the field redefinitions in eq. (8.6) connect a unitary non-power-counting renormalizable theory to a power-counting renormalizable one.

8.4 The Equivalence Theorem and Proca's gauge

We now want to understand what happens at quantum level, i.e. we apply the Equivalence Theorem to the transformations in eq. (8.6). We introduce a parameter ρ controlling the field redefinitions in eq. (8.6) and rewrite them as

$$\begin{aligned} A_\nu &\to A'_\nu \equiv A_\nu + \rho \frac{\sqrt{\alpha}}{m} \partial_\nu \varphi, \\ \psi &\to \psi' \equiv \exp\left(ie\rho \frac{\sqrt{\alpha}}{m} \varphi \right) \psi, \\ \bar{\psi} &\to \bar{\psi}' \equiv \exp\left(-ie\rho \frac{\sqrt{\alpha}}{m} \varphi \right) \bar{\psi}. \end{aligned} \tag{8.10}$$

Since the jacobian of the above field redefinition is one and the transformation is locally invertible, we can apply the WET.

The ET BRST transformation of the parameter ρ is provided by

$$s\rho = \theta, \qquad s\theta = 0, \tag{8.11}$$

where θ is an anti-commuting parameter. The FP-charge of ρ, θ is defined by setting $FP(\rho) = 0$, $FP(\theta) = 1$.

According to the formulation developed in section 2 for the WET we impose that A'_μ, ψ', $\bar\psi'$ are BRST invariant (see eq. (2.4)). In this way we obtain the BRST transformations of A_μ, ψ, $\bar\psi$ (denoted by $sA_\mu = \omega_\mu$, $s\psi = \omega$, $s\bar\psi = \bar\omega$), in agreement with eq. (2.7):

$$\begin{aligned} sA'_\mu &= \omega_\mu + \theta\frac{\sqrt{\alpha}}{m}\partial_\mu\varphi = 0\,, \\ s\psi' &= ie\theta\frac{\sqrt{\alpha}}{m}\varphi \exp\left(ie\rho\frac{\sqrt{\alpha}}{m}\varphi\right)\psi + \exp\left(ie\rho\frac{\sqrt{\alpha}}{m}\varphi\right)\omega = 0\,, \\ s\bar\psi' &= -i\bar\psi\exp\left(-ie\rho\frac{\sqrt{\alpha}}{m}\varphi\right)e\theta\frac{\sqrt{\alpha}}{m}\varphi + \bar\omega\exp\left(-ie\rho\frac{\sqrt{\alpha}}{m}\varphi\right) = 0\,. \end{aligned} \tag{8.12}$$

The solution for ω_μ is

$$\omega_\mu = -\theta\frac{\sqrt{\alpha}}{m}\partial_\mu\varphi \tag{8.13}$$

which is linear in the fields and does not require the introduction of any external source.

Notice that the solution for ω and $\bar\omega$

$$\omega = -ie\theta\frac{\sqrt{\alpha}}{m}\varphi\psi \qquad \text{and} \qquad \bar\omega = ie\bar\psi\theta\frac{\sqrt{\alpha}}{m}\varphi \tag{8.14}$$

is quadratic in the fields. Since the field φ is not interacting, according to eq. (8.4), the products of fields in eq. (8.14) are well-defined and do not require the introduction of the antifields for $\psi, \bar\psi$. Thus we can write the ET STI as:

$$\mathcal{S}(\Gamma) = \int d^4x \left(-\theta\frac{\sqrt{\alpha}}{m}\partial_\mu\varphi\frac{\delta\Gamma}{\delta A_\mu} - ie\theta\frac{\sqrt{\alpha}}{m}\varphi\psi\frac{\delta\Gamma}{\delta\psi} + ie\bar\psi\theta\frac{\sqrt{\alpha}}{m}\varphi\frac{\delta\Gamma}{\delta\bar\psi}\right) + \theta\frac{\delta\Gamma}{\delta\rho} = 0\,. \tag{8.15}$$

We wish to comment on the above equation. Eq. (8.15) looks like a Ward identity taking into account the gauge transformation of A_μ, ψ and $\bar\psi$. The last term of ET STI allows to control the dependence of all Green functions on the gauge parameter ρ. Eq. (8.15) guarantees that the Green functions of gauge-invariant operators are independent of ρ and thus we can look at Proca's gauge as the limit of ρ going to zero.

9. Concluding remarks

In this note we discussed the Equivalence Theorem (ET) in the BRST formalism.

Some discussion has been devoted to the kind of transformation of the fields: if the transformation has a local inverse (at least perturbatively) then some short-cut in the proof of the ET at quantum level can be used, but also if the inverse is non-local we proposed some alternative which allow the loop expansion both in the original and in the transformed theory.

We studied the quantum deformation to the associated ST identity and showed that suitably defined sets of Green functions do not depend on the choice of the transformation of the fields. The computation of the cohomology for the linearized ST identity has been performed by purely algebraic methods and no power-counting arguments have been used. The results of the paper are based on the conjecture of the Quasi Classical Action Principle.

As an example we studied the massive QED in Proca's gauge by relating it to a renormalizable gauge theory in the Lorentz-covariant gauge via the Equivalence Theorem.

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A. The Quasi Classical Action Principle

In the framework of Algebraic Renormalization an essential rôle is played by a set of locality properties of the renormalized Green functions encoded in the so-called Quantum Action Principle (QAP). The relevance of the QAP in discussing many aspects of the renormalized theory by purely algebraic and regularization-independent methods has emerged since the first pioneering papers on the BRST quantization of gauge theories. It is now believed that the QAP expresses the fundamental locality requirements a regularization scheme should possess in order to give rise to a sensible (local) power-counting renormalizable quantum field theory. These locality requirements are common to all regularization schemes and do not depend on the normalization conditions chosen.

Despite this kind of universality, the proof of the QAP has to be given independently for each regularization procedure. In the seventies the QAP has been proven for a variety of regularization schemes, including BPHZ and analytical and dimensional regularization. We wish to recall here the quite noticeable exception of the Epstein-Glazer formalism, for which no satisfactory proof of the QAP has been worked out up to now.

It is customary to summarize the content of the QAP by characterizing the behavior of the renormalized quantum effective action Γ under infinitesimal variations of the fields and the parameters of the model. The standard formulation of the QAP (as given for instance in [29]) is the following.

Proposition A.1 *Let Γ be the vertex functional corresponding to a (power-counting renormalizable) theory in a D-dimensional space-time with a classical action given by*

$$\Gamma^{(0)} = \int d^D x \, \mathcal{L}(\varphi_a, \beta_i, \lambda) \, , \tag{A.1}$$

where φ_a are the quantum fields, β_i the external sources coupled to field polynomials Q^i and λ stands for the parameters of the model (masses, coupling constants, renormalization points). Let the inverse of the quadratic part of the action be the standard Feynman propagators.

Given the local operator

$$\mathfrak{s}(\Gamma) \equiv \alpha_a \frac{\delta\Gamma}{\delta\varphi_a(x)} + \alpha_{ab}\varphi_b(x)\frac{\delta\Gamma}{\delta\varphi_a(x)} + \alpha_{ia}\frac{\delta\Gamma}{\delta\beta_i(x)}\frac{\delta\Gamma}{\delta\varphi_a(x)} + \alpha\frac{\delta\Gamma}{\delta\lambda} \, , \tag{A.2}$$

where $\alpha_a, \alpha_{ab}, \alpha_{ia}$ and α are constants, then the Quantum Action Principle can be stated in the following way

$$\mathfrak{s}(\Gamma) = \Delta(x) \cdot \Gamma = \Delta^{(n)}(x) + O(\hbar^{n+1}) \, . \tag{A.3}$$

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$\Delta(x) \cdot \Gamma$ denotes the insertion of a local operator. Moreover the lowest non-vanishing order coefficient $\Delta^{(n)}(x)$ of $\Delta(x) \cdot \Gamma$ is a local polynomial in the fields and external sources and their derivatives with bounded dimension.

At the integrated level (Slavnov-Taylor-like identities) the QAP reads

$$\mathcal{S}(\Gamma) \equiv \int d^4x \mathfrak{s}(\Gamma) = \int d^4x \Delta(x) \cdot \Gamma \,. \tag{A.4}$$

The first non-vanishing order of the ST-like breaking terms is given by

$$\Delta^{(n)} \equiv \mathcal{S}(\Gamma)^{(n)} = \int d^4x \Delta^{(n)}(x) \,. \tag{A.5}$$

As a consequence of proposition A.1, $\Delta^{(n)}$ is an integrated local polynomial in the fields and the external sources and their derivatives with bounded dimension.

The ultraviolet (UV) dimension d_Δ of $\Delta^{(n)}$ can be predicted from the UV dimensions d_a of the fields φ_a and the UV dimensions d_{Q^i} of the field polynomials Q^i [29]. We do not dwell on this problem here since the only information we need for the present discussion is the fact that d_Δ is bounded.

The QAP tells us that in a power-counting renormalizable theory the ST-like identity in eq. (A.4) can be broken at quantum level only by the insertion of an integrated local composite operator of bounded dimension. This is an all-order statement holding true regardless the normalization conditions chosen. At the lowest non-vanishing order the insertion $\Delta(x) \cdot \Gamma$ reduces to a local polynomial in the fields and external sources and their derivatives with bounded dimensions. This property is a consequence of the topological nature of the $\hbar$-expansion as a loop expansion. That is, if a local insertion in the vertex functional were zero up to the order $n-1$, at the n-th order it must reduce from a diagrammatic point of view to a set of points. By power-counting this set is finite and hence it corresponds to a local polynomial in the fields and the external sources and their derivatives.

The extension of the QAP beyond the power-counting renormalizable case is yet an open issue in the theory of renormalization. For a non-power-counting (but still polynomial) interaction lagrangian it is known in analytical renormalization that the breaking of the ST-like identities is given by the insertion of a (possibly infinite) sum of local operators. Quite restrictive assumptions on the form of the propagators are needed in the proofs of both the QAP and extensions thereof. In all cases the key ingredient of these diagrammatic analyses is Zimmerman's forest formula, allowing to treat in a systematic way the rather formidable combinatorial intricacies of the renormalization procedure.

A complementary approach has been recently formulated in [16] by Stora. A statement weaker than the QAP has been conjectured to hold true in very general situations, as a consequence of first principles of (axiomatic) local quantum field theory. Unlike in the QAP no reference is made to the all-orders characterization of ST-like breaking terms. It is only stated that the first non-vanishing order in the loop expansion is a local functional. Here locality should be generally understood in the sense of local formal power series, since the loss of power-counting forces to drop bounds on the dimensions.

The starting point is the existence of an off-shell S-matrix fulfilling causal factorization. We also assume that the free fields can be derived from a lagrangian, which in turn implies (in the presence of a mass gap) the validity of the LSZ reduction formulae. The latter allow to make the connection with the usual Green functions by defining their generating functional $Z[J,g]$ to be equal to the vacuum expectation value of the off-shell S-matrix. Here J is a collective notation for the external sources linearly coupled to the quantized fields $\{\varphi_a\}$ and g stands for all the external sources β coupled to composite operators and for all the coupling constants λ (promoted to test functions in the spirit of the Epstein-Glazer approach).

Stora's conjecture can the be summarized as follows [16]:

Proposition A.2 *Let*

$$\mathcal{D}(\epsilon) = \int d^4\, x\, \epsilon(x) \mathcal{D}_x[J,g]\,,$$

where $\mathcal{D}_x[J,g]$ is a local functional first order linear differential operator:

$$\mathcal{D}_x[J,g] = \mathcal{G}(J,g)(x)\frac{\delta}{\delta g(x)} + K(J,g)(x)\frac{\delta}{\delta J(x)}$$

with $\mathcal{G}$ and K are local in g and J, and $\epsilon(x)$ is a test function.

Let

$$W[J,g] = \sum_n \hbar^n W^{(n)}[J,g]$$

be the connected generating functional expanded in powers of the loop-counting parameter $\hbar$.

Assume that

$$\mathcal{D}(\epsilon) W^{(i)}[J,g] = 0 \qquad for \qquad i < n\,.$$

Then

$$\mathcal{D}(\epsilon) W^{(n)}[J,g] = \Delta^{(n)}[J,g,\varphi](\epsilon)\Big|_{\frac{\delta W^{(0)}[J,g]}{\delta J(x)} = \varphi(x)}$$

for some $\Delta^{(n)}$ local in (J,g,φ).

In this approach the diagrammatic analysis is avoided and no mention is made to the detailed structure of the interaction lagrangian of ordinary perturbative quantum field theory. Moreover in this formulation there is no reference to the concrete procedure allowing to construct the n-th order in the loop expansion, starting from lower orders. In particular we do not rest on any specific regularization procedure. In this sense the conjecture captures the universal features of the locality properties one should demand for any physically sensible renormalization. This in turn renders the content of the conjecture the natural expectation to be required for all possible extensions of ordinary quantum field theory to those situations where power-counting renormalizability might not hold.

Up to now no proof of proposition A.2 is known. In the present paper we assume its validity and make extensive use of the following consequence of proposition A.2, known as the Quasi-Classical Action Principle:

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Proposition A.3 *Let Γ be the vertex functional corresponding to a theory with a classical action given by*

$$\Gamma^{(0)} = \int d^4x \, \mathcal{L}(\varphi_a, \beta_i, \lambda) \,,$$

where φ_a are the quantum fields, β_i the external sources coupled to field polynomials Q^i, and λ stands for the parameters of the model (masses, coupling constants, renormalization points). We assume that the Legendre transform $W[J,\beta,\lambda]$ of $\Gamma[\varphi,\beta,\lambda]$

$$W[J,\beta,\lambda] = \Gamma[\varphi,\beta,\lambda] + \int d^4x \, J(x)\varphi(x) \,, \qquad J(x) = -\frac{\delta\Gamma}{\delta\varphi(x)} \,, \qquad \frac{\delta\Gamma}{\delta\beta(x)} = \frac{\delta W}{\delta\beta(x)} \,, \tag{A.6}$$

fulfills proposition A.2. Notice that $\Gamma^{(0)}$ is not required to be power-counting renormalizable. Let us also define

$$\mathfrak{s}(\Gamma) = \alpha_a \frac{\delta\Gamma}{\delta\varphi_a(x)} + \alpha_{ab}\varphi_b(x)\frac{\delta\Gamma}{\delta\varphi_a(x)} + \alpha_{ia}\frac{\delta\Gamma}{\delta\beta_i(x)}\frac{\delta\Gamma}{\delta\varphi_a(x)} + \alpha\frac{\delta\Gamma}{\delta\lambda} \,. \tag{A.7}$$

Then (Quasi Classical Action Principle) *the first non-vanishing order in the loop expansion, say n, of $\mathfrak{s}(\Gamma)$:*

$$\Delta^{(n)} \equiv \mathfrak{s}(\Gamma)^{(n)} \,, \qquad \mathfrak{s}(\Gamma)^{(j)} = 0 \qquad \textit{for } j = 0, 1, \ldots, n-1 \tag{A.8}$$

is a local formal power series in the fields and external sources and their derivatives.

In the power-counting renormalizable case bounds on the dimensions can be given truncating the formal power series predicted by the QCAP to a local polynomial. Thus the QCAP reduces in this case to the part of the QAP stating that the lowest non-vanishing order $\Delta^{(n)}(x)$ of $\Delta(x) \cdot \Gamma$ in eq. (A.3) is a polynomial. This justifies the name of Quasi-Classical Action Principle.

For most practical purposes the QCAP (or, for power-counting renormalizable theories, the part of the QAP relevant to $\Delta^{(n)}(x)$) is what is really needed in order to carry out the program of Algebraic Renormalization. In particular, this is enough to discuss the restoration of anomaly-free ST-like identities order by order in the loop expansion. This point is illustrated in the present paper on the example of the quantization of the Equivalence Theorem ST identities.

B. Cohomology of coupled doublets

The computation of the cohomology of an arbitray nilpotent differential s (including BRST differentials and classical linearized Slavnov-Taylor operators) can be simplified thanks to the elimination of the fields and external sources entering as doublets under s.

We make here the distinction between (decoupled) doublets, in the ordinary sense used in literature, and coupled doublets.

Definition B.1 *A pair of variables (ρ,θ) is called a (decoupled) doublet under the nilpotent differential s if and only if*

$$s\rho = \theta \,, \quad s\theta = 0 \tag{B.1}$$

and their counting operator

$$\mathcal{N} = \sum_k \int d^4x \left(\rho_k \frac{\delta}{\delta \rho_k} + \theta_k \frac{\delta}{\delta \theta_k} \right) \tag{B.2}$$

commutes with s:

$$[\mathcal{N}, s] = 0 \,. \tag{B.3}$$

The last condition justifies the name of decoupled doublets. Usually in the literature the variables fulfilling the conditions in eqs. (B.1) and (B.3) are simply called "doublets". Here we wish to charcterize them further as "decoupled" since we will drop soon the condition in eq. (B.3). It is a well-known fact that

Lemma B.1 *If a pair of variables* (ρ,θ) *is a decoupled doublet under the nilpotent differential* s *and* $\mathcal{I}[\rho,\theta,\zeta]$ *is a* s*-invariant local integrated formal power series depending on* ρ,θ *and on other variables collectively denoted by* ζ*, then its* (ρ,θ)*-dependent part is* s*-closed:*

$$\mathcal{I}[\rho,\theta,\zeta] - \mathcal{I}[0,0,\zeta] = s\mathcal{G}[\rho,\theta,\zeta]$$

for some local integrated formal power series $\mathcal{G}[\rho,\theta,\zeta]$.

Proofs of the above Lemma can be found for instance in [29] and in [34]. As a consequence of lemma B.1, it can be proven that the cohomology of s in the space of local integrated formal power series does not depend on (ρ,θ) [29, 34].

The external sources (ρ,θ), introduced in the discussion of the on-shell formulation of the Equivalence Theorem, fulfill eq. (B.1), where the relevant nilpotent differential s is now the classical linearized ST operator (see eq. (2.15) for the WET and (6.28) for the SET). However they do not meet eq. (B.3). We introduce then the following definition:

Definition B.2 *A pair of variables* (ρ,θ) *is called a "coupled doublet" under the nilpotent differential* s *if*

$$s\rho = \theta \,, \quad s\theta = 0$$

and their counting operator

$$\mathcal{N} = \sum_k \int d^4x \left(\rho_k \frac{\delta}{\delta \rho_k} + \theta_k \frac{\delta}{\delta \theta_k} \right) \tag{B.4}$$

does not commute with s

$$[\mathcal{N}, s] \neq 0 \,. \tag{B.5}$$

It turns out [31] that the cohomology of s in the space of local integrated formal power series can be completely characterized in terms of the doublets-independent component of s. However the statement of lemma B.1 is no more true. A weaker statement, which proves to be sufficient for the discussion of the ET STI in the on-shell formalism, can be proven [31]:

Lemma B.2 *Let $\mathcal{I}[\rho, \theta, \zeta]$ be a local integrated formal power series closed under the nilpotent differential s:*

$$s\mathcal{I} = 0\,. \tag{B.6}$$

Moreover let us assume that $s\mathcal{I}[0, 0, \zeta] = 0$. Then we have

$$\mathcal{I}[\rho, \theta, \zeta] - \mathcal{I}[0, 0, \zeta] = s\mathcal{G}[\rho, \theta, \zeta] \tag{B.7}$$

for some local integrated formal power series $\mathcal{G}[\rho, \theta, \zeta]$.

In particular we remark that eq. (B.6) is satisfied if

$$\mathcal{I}[0, 0, \zeta] = 0\,. \tag{B.8}$$

In the ET STI the n-th order ST breaking term $\Delta^{(n)}$ plays the rôle of $\mathcal{I}[\rho, \theta, \zeta]$. $\Delta^{(n)}$ has FP-charge +1. Since the only fields and external sources with FP-charge +1 all enter as doublets under the linearized classical ST operator, $\Delta^{(n)}$ fulfills eq. (B.8). Therefore we can apply lemma B.2. This in turn allows to prove that the on-shell ET STI are non-anomalous, as is explained in sections 2.2 and 6.1.

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